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STRONG-NORMALIZATION FOR ARITHMETIC
(VARIATIONS ON A THEME OF PRAWITZ)

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Strong-normalization for arithmetic (variations on a theme of Prawitz) *)

by

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ABSTRACT

We present a variant of PRAWITZ's proof of strong-normalization for GENTZEN's natural-deduction system for arithmetic, where inductive-definitions are replaced by explicit definitions.

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*)

This paper is not for review; it is meant for publication in a journal.

We refer to GENTZEN's natural-deduction system for arithmetic (GENTZEN [36] §5, PRAWITZ [71] III.1), for which we give a variant of PRAWITZ's ([71] app.A) proof of strong-normalization.

The main point of departure of this note from PRAWITZ's treatment is this: we define (an analogue to) "strong validity" (called below "stability") explicitly in arithmetic, instead of giving an inductive definition.

The *raisons d'être* of this variant seem to be:

1. It is an alternative, which some people might like.
2. The formatization of restricted versions of the results within arithmetic is direct (compared to TROELSTRA [73] IV.5,I.4).
3. A more economical measure of complexity on formulae, for the definition of "strong validity" is obtained (2.1).
4. This kind of treatment is applicable to infinitary derivations, where inductive definitions seem to fail altogether. (This matter will be treated in detail elsewhere).

In section 6 we indicate an even simpler variant of the proof, but a variant which works only for the disjunction-free fragment (and for which 4. fails).

1. REDUCTION-STEPS; STRONG NORMALIZABILITY

The reductions of 1.1-1.3 are defined by PRAWITZ ([65],[71])

1.1. Detour-reductions

(i)
$$\begin{matrix} \dfrac{\begin{matrix}\Sigma_0\\ A_0\end{matrix}\qquad \begin{matrix}\Sigma_1\\ A_1\end{matrix}}{A_0\&A_1} \\ A_i \end{matrix} \quad > \quad \begin{matrix}\Sigma_i\\ A_i\end{matrix} \qquad (i=0,1)$$

(ii)
$$\dfrac{\dfrac{\begin{matrix}[A]\\ \Sigma\\ B\end{matrix}}{A\to B}\qquad \begin{matrix}\Delta\\ A\end{matrix}}{B} \quad > \quad \begin{matrix}\Delta\\ [A]\\ \Sigma\\ B\end{matrix}$$

(iii) $$\dfrac{\begin{matrix}\Sigma \\ A_i \\ A_0 \vee A_1\end{matrix} \quad \begin{matrix}[A_0] \\ \Delta_0 \\ C\end{matrix} \quad \begin{matrix}[A_1] \\ \Delta_1 \\ C\end{matrix}}{C} \quad > \quad \begin{matrix}\Sigma \\ [A_i] \\ \Delta_i \\ C\end{matrix} \qquad (i=0,1)$$

(iv) $$\begin{matrix}\Sigma(a) \\ Aa \\ \forall x A x \\ At\end{matrix} \quad > \quad \begin{matrix}\Sigma(t) \\ At\end{matrix}$$

(v) $$\dfrac{\begin{matrix}\Sigma \\ At \\ \exists x A x\end{matrix} \quad \begin{matrix}[Aa] \\ \Delta(a) \\ B\end{matrix}}{B} \quad > \quad \begin{matrix}\Sigma \\ [At] \\ \Delta(t) \\ B\end{matrix}$$

1.2. Induction-reductions

(i) $$\dfrac{\begin{matrix}\Sigma \\ Ao\end{matrix} \quad \begin{matrix}Aa \\ \Delta(a) \\ A(Sa)\end{matrix}}{Ao} \quad > \quad \begin{matrix}\Sigma \\ Ao\end{matrix}$$

(ii) $$\dfrac{\begin{matrix}\Sigma \\ Ao\end{matrix} \quad \begin{matrix}Aa \\ \Delta(a) \\ A(Sa)\end{matrix}}{A(St)} \quad > \quad \begin{matrix}\dfrac{\begin{matrix}\Sigma \\ Ao\end{matrix} \quad \begin{matrix}Aa \\ \Delta(a) \\ A(Sa)\end{matrix}}{At} \\ \Delta(t) \\ A(St)\end{matrix}$$

1.3. Permutative reductions

(i)
$$\dfrac{\dfrac{\Sigma \quad \overset{\Delta}{A}}{A}\,\exists E \qquad (\Gamma_i)_i}{B}\,\rho \quad \succ \quad \dfrac{\Sigma \qquad \dfrac{\overset{\Delta}{A} \quad (\Gamma_i)_i}{B}\,\rho}{B}\,\exists E$$

(ii)
$$\dfrac{\dfrac{\Sigma \quad \overset{\Delta_1}{A} \quad \overset{\Delta_2}{A}}{A}\,\vee E \qquad (\Gamma_i)_i}{B}\,\rho \quad \succ \quad \dfrac{\Sigma \qquad \dfrac{\overset{\Delta_1}{A} \quad (\Gamma_i)_i}{B}\,\rho \qquad \dfrac{\overset{\Delta_2}{A} \quad (\Gamma_i)_i}{B}\,\rho}{B}\,\vee E$$

where ρ is an elimination-rule.

1.4. Semi-proper reductions

(1)
$$\dfrac{\overset{\Sigma}{A_0 \vee A_1} \qquad \begin{matrix}[A_0]\\ \Delta_0 \\ B\end{matrix} \qquad \begin{matrix}[A_1]\\ \Delta_1 \\ B\end{matrix}}{B} \quad \succ \quad \begin{matrix}[A_i]\\ \Delta_i \\ B\end{matrix} \qquad (i=0,1)$$

(2)
$$\dfrac{\overset{\Sigma}{\exists x A x} \qquad \begin{matrix}[Aa]\\ \Delta \\ B\end{matrix}}{B} \quad \succ \quad \begin{matrix}[Aa]\\ \Delta \\ B\end{matrix}$$

(These reductions preserve the derived formula, but alter the set of open assumptions, except if the indicated set of discharged assumptions is empty, in which case they are identical with PRAWITZ's [71] 3.3.2 immediate-simplifications).

1.5. Inner-reductions

If Δ is a proper subderivation of Σ', $\Delta > \Delta'$ by one of the above, and Σ' comes from Σ by replacing Δ with Δ', then $\Sigma \succ \Sigma'$. We say then that $\Sigma > \Sigma'$ by an *inner-reduction*; $\Delta \succ \Delta'$ by 1.1-1.4 we call a *main-reduction*.

1.6. Δ is *strongly normalizable* (s.n.), if there is natural-number n such that $\Delta > \Delta_1 \succ \Delta_2 \succ \dots \succ \Delta_n$ is impossible. If Δ is strongly-normalizable we write $\nu(\Delta)$ for the minimal n satisfying the above condition.

1.7. Remark. The treatment below may be modified to apply to a more general definition of permutative-reductions, where ρ is allowed to be any inference-rule except induction (for the case of $\rightarrow$I such a reduction may, however, alter the set of open assumptions of the derivation). For applications of the strong-normalization theorem, however, this generalization is superfluous. We therefore prefer to treat the restricted definition, allowing a greater clarity of the proofs.

2. IMPROPER REDUCTIONS, STABILITY

2.1. A measure of complexity

The measure μ on formulae is defined by recursion on their length:

$\mu(A) := o$ for A atomic
$\mu(A\&B) :\equiv \mu(A \vee B) := \max[\mu(A),\mu(B)]$
$\mu(\forall xAx) :\equiv \mu(\exists xAx) := \mu(A\bar{o})$
$\mu(A \rightarrow B) := \max[\mu(A),\mu(B)] + 1.$

For a derivation Δ with a derived formula A we also write $\mu(\Delta) := \mu(A)$.

2.2. Improper-reductions

Assume that the notion "stability" and the reduction-step $\succ_o$ are defined for derivations Δ such that $\mu(\Delta) < n$. For Δ s.t. $\mu(\Delta) = n$ we then define

(i) $$\frac{\begin{matrix}\Sigma_0 & \Sigma_1\\ A_0 & A_1\end{matrix}}{A_0 \& A_1} \quad \succ\!\circ \quad \begin{matrix}\Sigma_i\\ A_i\end{matrix} \qquad (i=0,1)$$

(ii) $$\frac{\begin{matrix}[A]\\ \Sigma\\ B\end{matrix}}{A \to B} \quad \succ\!\circ \quad \begin{matrix}\Delta\\ [A]\\ \Sigma\\ B\end{matrix} \qquad \text{whenever } \begin{matrix}\Delta\\ A\end{matrix} \text{ is stable}$$

(iii) $$\frac{\begin{matrix}\Sigma(a)\\ Aa\end{matrix}}{\forall x Ax} \quad \succ\!\circ \quad \begin{matrix}\Sigma(t)\\ At\end{matrix} \qquad \text{for every term } t$$

(iv) $$\frac{\begin{matrix}\Sigma\\ A_i\end{matrix}}{A_0 \vee A_1} \quad \succ\!\circ \quad \begin{matrix}\Sigma\\ A_i\end{matrix} \qquad (i=0,1)$$

(v) $$\frac{\begin{matrix}\Sigma\\ At\end{matrix}}{\exists x Ax} \quad \succ\!\circ \quad \begin{matrix}\Sigma\\ At\end{matrix}$$

Note that these reductions do not preserve the meaning of derivations. They only have a combinatorial role in the proof of strong-normalization.

2.4. We write $\Delta \gg \Delta'$ if for some $n \geq 0$ $\Delta = \Delta_0 \succ\!\!\!\prec \Delta_1 \succ\!\!\!\prec \ldots \succ\!\!\!\prec \Delta_n = \Delta'$, where $\succ\!\!\!\prec$ is either $\succ$ or $\succ\!\circ$.

2.5. Stability

It is seen outright that if $\Delta \succ\!\!\!\prec \Delta'$ then $\mu(\Delta') \leq \mu(\Delta)$. Hence the definition of $\Delta \gg \Delta'$ uses the notion of stability only for derivations Γ s.t. $\mu(\Gamma) < \mu(\Delta)$. So we may define: Δ is *stable* if $\Delta \gg \Delta' \Rightarrow \Delta'$ is s.n..

2.6. We write $\Delta \mapsto \Delta^*$ if Δ^* is obtained from Δ by substituting terms for parameters free in Δ and then substituting stable derivations for some open assumptions.

2.7. Δ is stable under substitution (s.s.) if $\Delta \mapsto \Delta^*$ implies Δ^* is stable.

2.8. Lemma. *Δ is stable iff $\Delta \succ \Delta'$ implies Δ' is stable.*

2.9. Lemma. *Every s.s. derivation is stable, and every stable derivation is s.n..*

2.10. Lemma. *If Δ' is a subderivation of Δ, and Δ is s.n. then Δ' is s.n. and $\nu(\Delta') \leq \nu(\Delta)$.*

2.8-2.10 are immediate from the definitions.

3. TREATMENT OF INTRODUCTION-INFERENCES AND INDUCTION

3.1. Proposition. *If $\Delta \equiv \dfrac{\Delta_0(\Delta_1)}{A}\rho$ where ρ is an introduction-rule and Δ_0 (and Δ_1) are s.s., then Δ is s.s..*

Proof. By 3.3, 3.7, 3.11 and 3.12 below. □

3.2. Lemma. *If $\begin{matrix}\Delta_0\\ A\end{matrix}, \begin{matrix}\Delta_1\\ B\end{matrix}$ are stable then so is*

$$\Delta \equiv \dfrac{\begin{matrix}\Delta_0\\ A\end{matrix} \quad \begin{matrix}\Delta_1\\ B\end{matrix}}{A\&B}$$

Proof. By induction on $\nu(\Delta_0) + \nu(\Delta_1)$. If $\Delta \succ \Delta'$ then this reduction is necessarily an inner one,

$$\Delta' \equiv \dfrac{\begin{matrix}\Delta_0'\\ A\end{matrix} \quad \begin{matrix}\Delta_1'\\ B\end{matrix}}{A\&B}$$

where $\nu(\Delta_0')+\nu(\Delta_1') < \nu(\Delta_0)+\nu(\Delta_1)$, hence Δ' is stable by induction hypothesis. If $\Delta \succ_0 \Delta' \equiv \begin{matrix}\Delta_0\\ A\end{matrix}$, say, then Δ' is stable by assumption. By 2.8 Δ is stable. □

3.3. Lemma. *If Δ_0, Δ_1 of 3.2 are s.s., then so is Δ.*

Proof. Immediate from 3.2. □

3.4. Definition. Let $\begin{matrix}[A]\\ \Delta\end{matrix}$ be a derivation, where $[A]$ is a set of open assumptions of Δ of the form A. We say that Δ is *strongly stable at* $[A]$ if for every stable derivation $\begin{matrix}\Gamma\\ A\end{matrix}$, $\begin{matrix}\Gamma\\ [A]\\ \Delta\end{matrix}$ is stable.

3.5. Lemma. *Let* $\begin{matrix}[A]\\ \Delta\end{matrix}$ *be s.s. at* $[A]$, $\begin{matrix}[A]\\ \Delta\end{matrix} \succ \begin{matrix}[A]'\\ \Delta'\end{matrix}$ *where* $[A]'$ *is the set of copies of elements of* $[A]$. *Then* Δ' *is s.s. at* $[A]'$.

Proof. Immediate by induction on $\nu(\Delta)$.(Note that the same Γ is substituted for every $\underline{A} \in [A]$ in 3.2, and that no assumption of Γ may be discharged in Δ in $\begin{matrix}\Gamma\\ [A]\\ \Delta\end{matrix}$). □

3.6. Lemma. *If* $\begin{matrix}[A]\\ \Delta\\ B\end{matrix}$ *is s.s. at* $[A]$, *then*

$$\Sigma \equiv \cfrac{\begin{matrix}[A]\\ \Delta\\ B\end{matrix}}{A\to B}$$

is stable.

Proof. By induction on $\nu(\Delta)$. If $\Sigma \succ \Sigma' \equiv \dfrac{\Delta'}{A\to B}$ then $\nu(\Delta') < \nu(\Delta)$, Δ' satisfies the conditions of the lemma by 3.5, and we are done by ind. hyp.
If

$$\Sigma \succ\!\circ\; \Sigma' \equiv \begin{matrix}\Gamma\\ [A]\\ \Delta\\ B\end{matrix}$$

then Σ' is stable, since Δ is s.s. at $[A]$ by assumption.

Hence by 2.8 Σ is stable. □

3.7. *If* $\begin{matrix}[A]\\ \Delta\\ B\end{matrix}$ *is s.s., then*

$$\cfrac{\begin{matrix}[A]\\ \Delta\\ B\end{matrix}}{A\to B} \equiv \Sigma$$

is s.s..

Proof. Let

$$\Sigma \mapsto \Sigma^* \equiv \frac{\begin{matrix}[A^*] \\ \Delta^* \\ B^*\end{matrix}}{A^* \to B^*} .$$

Δ is s.s., hence Δ^* is s.s. at $[A^*]$, so by 3.6 Σ^* is stable. So Σ is s.s. □

3.8. Lemma. *If* a *is free in* $\Sigma(a)$, $\Sigma \succ \Sigma'$ *then* a *is free in* Σ', *and* $\Sigma(t) \succ \Sigma'(t)$ *for every term* t.

3.9. Lemma. *If* $\Delta \mapsto \Delta^*$ *and* a *does not occur in any open assumption of* Δ *then* $\Delta \mapsto \Delta^*[t/a]$ *for every term* t.

The proofs of 3.8 and 3.9 are immediate.

3.10. Lemma. *If* a *is free in* $\begin{matrix}\Delta(a) \\ Aa\end{matrix}$ *and* $\Delta(t)$ *is stable for every* t *then*

$$\Sigma \equiv \begin{matrix}\Delta(a) \\ Aa \\ \forall xAx\end{matrix}$$

(if at all a correct derivation) is stable.

Proof. By induction on $\nu(\Delta)$ (as in 3.2). If

$$\Sigma \succ \Sigma' \equiv \frac{\begin{matrix}\Delta'(a) \\ Aa\end{matrix}}{\forall xAx}$$

then $\nu(\Delta') < \nu(\Delta)$ and by 3.8 a is free in Δ' and $\Delta'(t)$ is stable for every t. Hence by the induction hypothesis Δ' is stable. If $\Sigma' \succ\!\!\circ\ \Sigma \equiv \Delta(t)$ then Σ' is stable outright by assumption. By 2.8 Σ is stable. □

3.11. Lemma. *If* $\begin{matrix}\Delta(a) \\ Aa\end{matrix}$ *is s.s. then so is*

$$\Sigma \equiv \frac{\begin{matrix}\Delta(a) \\ Aa\end{matrix}}{\forall xAx} .$$

Proof. Let

$$\Sigma \mapsto \Sigma^* \equiv \frac{\begin{matrix}\Delta^*(a)\\ A^*a\end{matrix}}{\forall x A^* x} .$$

By 3.9 $\Delta \mapsto \Delta^*(t)$ for every t, so $\Delta^*(t)$ is stable. By 3.10 then Σ^* is also stable, as required. □

3.12. Lemma.

(i) *If* $\begin{matrix}\Delta\\ At\end{matrix}$ *is s.s., then so is* $\frac{\begin{matrix}\Delta\\ At\end{matrix}}{\exists x Ax}$.

(ii) *If* $\begin{matrix}\Delta\\ A\end{matrix}$ *is s.s. then so are* $\frac{\begin{matrix}\Delta\\ A\end{matrix}}{A \vee B}$, $\frac{\begin{matrix}\Delta\\ A\end{matrix}}{B \vee A}$.

Proof. Similar to 3.2-3.3. □

3.13. Lemma. *If* $\begin{matrix}\Sigma\\ Ao\end{matrix}$ *is stable, and for every term* t $\begin{matrix}[At]\\ \Delta(t)\end{matrix}$ *is s.s. at* [At], *then*

$$\Pi \equiv \frac{\begin{matrix}\Sigma\\ Ao\end{matrix} \quad \begin{matrix}[Aa]\\ \Delta(a)\\ A(Sa)\end{matrix}}{A(t)} \text{IND}$$

is stable for every term t.

Proof. By induction on $\nu(\Sigma) + \nu(\Delta) + \tau(\Pi)$, where $\tau(\Pi)$ is defined as follows:

$\tau(t) := o$, if t is a term and for no term s $t = Ss$,

$\tau(St) := \tau(t) + 1$,

$\tau(\Pi) := \tau(t)$, if the main inference-rule of the derivation Π is IND, with t as a proper term,

$\tau(X) := o$ else.

Now if $\Pi > \Pi'$ by an inner reduction then $\tau(\Pi') = \tau(\Pi)$, $\nu(\Sigma')+\nu(\Delta') < \nu(\Sigma)+\nu(\Delta)$ and as in the proof of 3.10 Π' satisfies the assumptions of the lemma. Hence by the induction hypothesis Π' is stable.

If $t = 0$, $\Pi > \Pi' \equiv \Sigma$ then Π' is stable by assumption.

If

$$\Pi \succ \Pi' \equiv \left.\begin{array}{c} \\ \Sigma \\ \underline{Ao} \end{array} \begin{array}{c} [Aa] \\ \Delta(a) \\ \underline{A(Sa)} \\ [At] \\ \Delta(t) \end{array}\right\} \Pi_0 \quad \begin{array}{c} \\ \\ \\ \\ \\ A(St) \end{array}$$

then $\tau(\Pi_0) \prec \tau(\Pi)$, so by the induction hypothesis Π_0 is stable. By assumption $\Delta(t)$ is s.s. at [At], so Π' is stable. The lemma follows by 2.8. □

3.14. Proposition. *If* Σ, $\Delta(a)$ *of* 3.13 *are s.s. then so is* Π.

Proof. 3.14 follows 3.13 like 3.11 follows 3.10. □

4. TREATMENT OF ELIMINATION-INFERENCES

4.0. Notations and definitions.

For the sake of brevity we shall skip cases for disjunction-rules, which are to be treated in complete analogy to the ∃-rules. Let $\Pi^0(\Pi^1)$ denote the left (right) main subderivation of Π, and $\lambda(\Pi)$ denote the height of Π (as a tree).

If Π^0, Π^1 are s.n., we define for Π the measure $i(\Pi)$ by $i(\Pi) := \langle\nu(\Pi^0),\lambda(\Pi^0),\nu(\Pi^1)\rangle$.

Let $\Sigma \equiv \begin{array}{c}\Sigma\\ \exists xAx\end{array}$. $\begin{array}{c}[At]\\ \Delta\end{array}$ is *stable under* Σ *at* [At] if Δ is stable, and whenever $\Sigma \succ \ldots \succ \begin{array}{c}\Theta\\ \underline{At}\\ \exists xAx\end{array}$ then $\begin{array}{c}\Theta\\ [At]\\ \Delta\end{array}$ is stable.

4.1. Lemma. *If* $\begin{array}{c}[A]\\ \Delta\end{array}$ *is stable under* Σ *at* [A] *then*

(i) *if* $\begin{array}{c}[A]\\ \Delta\end{array} \succ \begin{array}{c}[A]'\\ \Delta'\end{array}$ *then* Δ' *is stable under* Σ *at* [A]';

(ii) *if* $\Sigma \succ \Sigma'$ *then* Δ *is stable under* Σ' *at* [A].

Proof. (i) is analogous to 3.5. (ii) is immediate from the definition. □

4.2. Main lemma. *Let* $\Pi \equiv \dfrac{\Pi^0(\Pi^1)}{A}\rho$ *be given s.t. either*

(i) ρ *is an elimination-rule other than* $\exists E$, *and* Π^0, Π^1 *are stable; or*

(ii) ρ *is* $\exists E$,

$$\Pi \equiv \dfrac{\begin{matrix}\Pi^0 \\ \exists xAx\end{matrix} \quad \begin{matrix}[A(a)] \\ \Pi^1(a) \\ B\end{matrix}}{B}$$

say, Π^0 *is s.n., and for every* t $\Pi^1(t)$ *is stable under* Π^0 *at* [At].

Then Π *is stable.*

Proof. By induction on $i(\Pi)$. I.e., we assume that every Θ satisfying the conditions of the lemma and $i(\Theta) < i(\Pi)$ is stable, and we prove that $\Pi \succ \Delta \Rightarrow \Delta$ is stable (which implies that Π is stable by 2.8).

Case [a]: $\Pi \succ \Delta$ by an inner reduction in Π^0. Then $\nu(\Delta^0) < \nu(\Pi^0)$ so $i(\Delta) < i(\Pi)$. If (i) applies to Π (and to Δ) then Δ satisfies the conditions of the lemma by 2.8, and if (ii) applies - by 4.1 (ii). So by the induction hypothesis Δ is stable.

Case [b]: $\Pi \succ \Delta$ by an inner reduction in Π^1. Then $\Delta^0 = \Pi^0$, $\nu(\Delta^1) < \nu(\Pi^1)$ so $i(\Delta) < i(\Pi)$. Δ satisfies the lemma's conditions by 2.8 - if (i) applies, and by 4.1 (i) - if (ii) applies. By the induction hypothesis Δ is stable.

Case [c]: (i) applies, and $\Pi \succ \Delta$ by a main direct reduction. Take the case $\rho = \to E$ (the argument is similar for &E and $\forall$E).

$$\Pi \equiv \Pi^0 \left\{ \dfrac{\dfrac{\begin{matrix}[A] \\ \Gamma \\ B\end{matrix}}{A \to B} \quad \begin{matrix}\Pi^1 \\ A\end{matrix}}{B} \right. \quad \succ \quad \begin{matrix}\Pi^1 \\ [A] \\ \Gamma \\ B\end{matrix} \quad \equiv \Delta$$

Π^0 and Π^1 are assumed stable, so

$$\Pi^0 \succ\!\circ \begin{array}{c}\Pi^1\\ [A]\\ \Gamma\\ B\end{array} \equiv \Delta$$

hence Δ is stable (2.8).

Case [d]: (ii) applies, and $\Pi \succ \Delta$ by a main direct reduction.

$$\Pi \equiv \dfrac{\begin{array}{c}\Pi^0\\ At\\ \hline \exists xAx\end{array} \quad \begin{array}{c}[Aa]\\ \Pi^1(a)\\ B\end{array}}{B} \succ \begin{array}{c}\Pi^0\\ [At]\\ \Pi^1(t)\\ B\end{array} \equiv \Delta .$$

By condition (ii) Δ is stable.

Case [e]: (i) applies and $\Pi \succ \Delta$ by a permutative reduction.

$$\Pi \equiv \Pi^0\left\{\dfrac{\dfrac{\begin{array}{c}\Gamma_0\\ \exists xAx\end{array} \quad \begin{array}{c}[Aa]\\ \Gamma_1(a)\\ B\end{array}}{B}\,\exists E \qquad (\Pi^1)}{C}\,\rho\right. \succ$$

$$\dfrac{\begin{array}{c}\Gamma_0\\ \exists xAx\end{array} \quad \dfrac{\begin{array}{c}[Aa]\\ \Gamma_1(a)\\ B\end{array} \quad (\Pi^1)}{C}\,\rho}{C}\,\exists E \equiv \Delta$$

Π^0 is stable by assumption, $\Pi^0 \succ \Gamma_1$ (by a semi-proper reduction, cf. 1.4), so Γ_1 is stable, and $\nu(\Gamma_1) < \nu(\Pi^0)$. Π^1 is stable by assumption, hence Δ^1 is stable. $\Delta^0 = \Gamma_0$ is a subderivation of Π^0, hence it is s.n. by 2.10, and $\nu(\Delta^0) \leq \nu(\Pi^0)$ while $\lambda(\Delta^0) < \lambda(\Pi^0)$. So $i(\Delta) < i(\Pi)$.

To show that Δ satisfies condition (ii) of the lemma, it remains to see that whenever

$$(*) \qquad \begin{array}{c}\Gamma_0\\ \exists xAx\end{array} \succ \cdots \succ \begin{array}{c}\Theta\\ At\\ \exists xAx\end{array}$$

then

$$\begin{array}{c}\Theta\\ [At]\\ \Gamma_1(t)\\ \dfrac{B \qquad (\Pi^1)}{C}\,\rho\end{array} \qquad =: \Xi$$

is stable.

But if (*), then

$$\Pi^0 \succ \dots \succ \begin{array}{c}\Theta\\ [At]\\ \Gamma_1(t)\\ B\end{array} \equiv \Xi^0 ;$$

so $\nu(\Xi^0) < \nu(\Pi^0)$ and $i(\Xi) < i(\Pi)$.

Π^0 is assumed stable, hence Ξ^0 is stable (2.8), while $\Xi^1 \equiv \Pi^1$ is assumed stable outright. Hence Ξ satisfies case (i) of the conditions of the lemma, and by the induction hypothesis Ξ is stable. Hence Δ satisfies case (ii) of the lemma's condition, and by the induction hypothesis Δ is stable.

<u>Case [f]</u>: (ii) applies, and $\Pi \succ \Delta$ by a permutative reduction.

$$\Pi \equiv \dfrac{\dfrac{\begin{array}{c}\Gamma_0\\ \exists xAx\end{array} \quad \begin{array}{c}(1)\\ [Aa]\\ \Gamma_1(a)\\ \exists yBy\end{array}}{\exists yBy}\,(1)\ \exists E \qquad \begin{array}{c}(2)\\ [Bb]\\ \Pi^1(b)\\ C\end{array}}{C}\,(2)\ \exists E \qquad \succ$$

$$\left.\dfrac{\begin{array}{c}\Gamma_0\\ \exists xAx\end{array} \quad \dfrac{\begin{array}{c}(1)\\ [Aa]\\ \Gamma_1(a)\\ \exists yBy\end{array} \quad \begin{array}{c}(2)\\ [Bb]\\ \Pi^1(b)\\ C\end{array}}{C}\,(2)\ \exists E}{C}\,(1)\ \exists E\right\}\ \Delta^1(a) \qquad \equiv \Delta .$$

Π^1 is stable at [Bb] under Π^0, $\Pi^0 \succ \Gamma_1$, hence (by 4.1 (i)) Π^1 is stable at [Bb] under Γ_1. We conclude that Δ^1 is stable, and that $i(\Delta) < i(\Pi)$ like in case [e].

It remains to show that for every t $\Delta^1(t)$ is stable at $\ulcorner At \urcorner$ under Γ_0; i.e., that if

$$(*) \qquad \Gamma_0 \succ \dots \succ \begin{matrix} \Theta \\ At \\ \exists xAx \end{matrix}$$

then

$$\Xi : \quad \dfrac{\begin{matrix} \Theta \\ [At] \\ \Gamma_1(t) \\ \exists yBy \end{matrix} \quad \begin{matrix} \\ [Bb] \\ \Pi^1(b) \\ C \end{matrix}}{C} \exists E$$

is stable. But, like in [e], (*) implies that $\Pi^0 \succ \dots \succ \Xi^0$, so Ξ^0 is s.n., $\nu(\Xi^0) < \nu(\Pi^0)$ and $i(\Xi) < i(\Pi)$.
($\nu(\Xi^1)$ is well-defined, because $\Xi^1 \equiv \Pi^1$ which is stable by assumption, so s.n.).

Ξ satisfies case (ii) of the lemma's conditions by 4.1 (ii), so by the induction hypothesis Ξ is stable. Hence Δ satisfies case (ii) of the lemma's conditions, and by the induction hypothesis Δ is stable.

Case [g]: (ii) applies and $\Pi \succ \Delta$ by a semi-proper reduction:

$$\frac{\Pi^0 \quad \Pi^1}{A} \exists E \succ \Pi^1 \equiv \Delta$$

then Δ is stable by assumption.

This concludes the proof. □

4.3. Corollary. *If* $\Pi \equiv \dfrac{\Pi^0 \quad (\Pi^1)}{A} \rho$ *where* ρ *is an elimination-inference, and* Π^0, Π^1 *are s.s., then* Π *is s.s..*

Proof. If ρ is &E, →E or ∀E this follows case (i) of 4.2 outright. If ρ is ∃E (or analogously - ∨E) then, if

$$\Pi \mapsto \Pi^* \equiv \cfrac{\begin{matrix} & [Ba] \\ \Pi^{0*} & \Pi^{1*}(a) \\ \exists x Bx & A \end{matrix}}{A}\ \exists E$$

then a does not occur in any open assumption of Π, so we have that

$$\Pi^1 \mapsto \begin{matrix} \Sigma \\ [Bt] \\ \Pi^{1*}(t) \end{matrix}$$

for any t and any stable $\begin{matrix}\Sigma\\ Bt\end{matrix}$. In particular, if

$$\Pi^{0*} \succ \dots \succ \cfrac{\begin{matrix}\Theta \\ Bt\end{matrix}}{\exists x Bx} \succ^{\circ} \begin{matrix}\Theta \\ Bt\end{matrix}$$

then $\begin{matrix}\Theta\\ At\end{matrix}$ is stable, since Π^{0*} is stable (by assumption). So $\begin{matrix}\Theta \\ [Bt] \\ \Pi^{1*}(t)\end{matrix}$ is stable, and the conditions of case (ii) of 4.2 are satisfied, hence Π^* is stable, as required. □

5. THE STRONG NORMALIZATION THEOREM

5.1. Theorem. *Every derivation* Π *is s.s.*.

Proof. By induction on $\lambda(\Pi)$. For $\lambda(\Pi) = 1$, i.e., Π is a singleton-derivation, the theorem is immediate from the definition of s.s..

If the main rule of Π is a Post-rule, the theorem follows trivially by the induction hypothesis applied to Π^0 (and Π^1).

If the main rule of Π is an introduction-rule or an induction, Π is s.s. by the induction hypothesis on Π^0 and Π^1 by 3.1, and if this is an elimination-rule - by 4.3. □

5.2. Corollary. (strong-normalization theorem). *Every derivation is s.n.*.

Proof. By 5.1 and 2.9. □

6. ANOTHER VARIANT OF THE NORMALIZATION-PROOF

6.0. For the disjunction-free fragment we may give an even simpler alternative proof, as below. Note that disjunction is eliminable in intuitioustic arithmetic (TROELSTRA [73] 1.3.7, LEIVANT [73] IV.1).

6.1. Define the reduction relation $\succ$ as follows.

(1) Detour-reductions - like 1.1.

(2) Detour-reductions through $\exists$E:

$$\dfrac{\Delta_k \qquad \begin{array}{c}\dfrac{\Delta_1 \qquad \dfrac{\Delta_0 \qquad \dfrac{\overset{\Sigma_0}{A} \quad \overset{\Sigma_1}{B}}{A\&B}}{A\&B}\,\exists E}{A\&B}\,\exists E\\ \vdots\end{array}}{\begin{array}{c}A\&B\\ A\end{array}}\,\exists E \qquad \succ \qquad \dfrac{\Delta_k \qquad \begin{array}{c}\dfrac{\Delta_1 \qquad \dfrac{\Delta_0 \qquad \overset{\Sigma_0}{A}}{A}\,\exists E}{A}\,\exists E\\ \vdots\end{array}}{A}\,\exists E$$

and similar clauses corresponding to the other detour-reductions.

(3) Induction-reductions - like 1.2.

(4) Inner-reductions - like 1.4.

The definition of strong-normalizability follows as in 1.

6.2. To the clauses of 2.2 add to the definition of $\succ\!\circ$ improper-reductions through $\exists$E:

$$\dfrac{\Delta_k \qquad \begin{array}{c}\dfrac{\Delta_0 \qquad \dfrac{\overset{\Sigma_0}{A} \quad \overset{\Sigma_1}{B}}{A\&B}}{A\&B}\,\exists E\\ \vdots\end{array}}{A\&B}\,\exists E \qquad \succ\!\circ \qquad \dfrac{\Delta_k \qquad \begin{array}{c}\dfrac{\Delta_0 \qquad \overset{\Sigma_0}{A}}{A}\,\exists E\\ \vdots\end{array}}{A}\,\exists E$$

etc.

The definitions of stability, and s.s. follow.

6.3. The treatment of the introduction-rules is now essentially the same as in 3, and the elimination-rules are even simpler (the inference cases are treated separately, and only a simple argument is to be added for the new reductions). Thus the strong-normalization theorem is obtained. Here we get all the corollaries of normalization without referring to permutative reductions, because of the presence of reductions through ∃E.

6.4. Permutative reductions of the most general kind (1.7) may be reinserted into the treatment without destroying its simplicity.

Let $\rho_1,\ldots,\rho_k$ enumerate the instances of ∃E in a given derivation Δ, and $\underline{A}_1,\ldots,\underline{A}_k$ their respective major-premises. Say that $\underline{A}_i$ is *subordinated to* $\underline{A}_j$ iff $\underline{A}_i$ occurs in the minor premise of ρ_j. Define

$$\sigma_\Delta(\underline{A}_i) := \underline{\max}[\sigma_\Delta(\underline{A}_j) \mid \underline{A}_i \text{ is subordinated to } \underline{A}_j]$$

$$h_\Delta(\underline{A}_i) := \text{the height of } \underline{A}_i \text{ in } \Delta$$

$$\mu_\exists(\Delta) := \langle n_1,\ldots,n_k\rangle \quad \text{where}$$

$$n_r := \sum\{h_\Delta(\underline{A}_i) \mid \sigma_\Delta(\underline{A}_i) = r\} \quad \text{(usual summation)} .$$

Then, if $\Delta \succ \Delta'$ by a permutative reduction, then $\mu_\exists(\Delta') < \mu_\exists(\Delta)$ and $\nu(\Delta') = \nu(\Delta)$.

Now, every derivation Δ is strongly normalizable relative reduction-sequences allowing permutative reductions (p.s.n. say), by induction on $\langle\nu(\Delta),\mu_\exists(\Delta)\rangle$, since

$$\Delta \succ \Delta' \text{ by a non-permutative reduction} \Rightarrow \nu(\Delta') < \nu(\Delta);$$

$$\Delta \succ \Delta' \text{ by a permutative reduction} \Rightarrow \nu(\Delta') = \nu(\Delta) \text{ and } \mu_\exists(\Delta') < \mu_\exists(\Delta).$$

7. FORMALIZATION OF THE PROOF IN ARITHMETIC.

7.1. The formalization within arithmetic of our proof of strong normalization is routine, except for one point: the arithmetization of the stability-predicate.

Let $\mathcal{St}_k(\Gamma)$ be a tentative abreviation for the formalization of " $\mu(\Gamma) \leq k$ and Γ is stable ". Strong normalizability is seen outright to be formalizable as a Σ^0_2 predicate ($\mathcal{Sn}$ say). Since $\mathcal{St}_0(\Delta) \leftrightarrow \mathcal{Sn}(\Delta)$ if $\mu(\Delta) = 0$, $\mathcal{St}_0$ is also a Σ^0_2 predicate.
If $\mu(\Delta) = n+1$ then $\Delta \succ\!\circ\, \Delta'$ is in general formalizable as a predicate of the form

$$(\exists\Gamma < \Delta')\,[\mathcal{St}_n(\Gamma)\ \&\ F(\Delta,\Delta',\Gamma)]$$

where F is a p.r. relation (and where Greek majuscules are used as variables for g.n.'s of derivations). $\Delta \succ\!\succ \Delta'$ is Σ^0_1 in $\succ$, hence $\Delta \succ\!\succ \Delta'$ is Σ^0_1 in $\mathcal{St}_n$.

$$\mathcal{St}_{n+1}(\Delta) \equiv \forall\Delta'[\Delta \succ\!\succ \Delta' \rightarrow \mathcal{Sn}(\Delta')],$$

so $\mathcal{St}_1(\Delta)$ is of the form $\forall[\Sigma^0_2 \rightarrow \Sigma^0_2]$ which is classically a Π^0_3-predicate; and for $n \geq 2$ we can see by induction that $\mathcal{St}_n$ is classically-equivalent to a Π^0_{n+2}-predicate.

7.2. Consequently, we may formalize within Π^0_{n+k}-arithmetic (where k is fixed for every n) the normalization-proof for all derivations Δ, satisfying: "if A occurs in Δ then $\mu(\Delta) \leq n$". Some consequences of this are given in TROELSTRA [73] IV.4.

7.3. Our proof of normalization illustrates the essential place of implication in formulae-complexity, since implication is the *only* logical symbol counted for the measure μ. By 7.2 normalization of derivations with a bound on the nesting of implications in the formulae (but with no bound on the alternations of quantifiers) is formalized within arithmetic.

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